



Uniformly constructing exact discrete soliton solutions and periodic solutions to differential–difference equations

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ABSTRACT

In this paper, with the aid of symbolic computation, we present a uniform method for constructing soliton solutions and periodic solutions to nonlinear differential–difference equations. And we successfully solve the famous mKdV lattice equation.

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1. Introduction

It is well known that nonlinear complex physical phenomena are related to non-linear evolution equations (NLEEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As they are mathematical models of the phenomena, the investigation of exact solutions of NLEEs plays an important role in the study of these phenomena. In the past few years, many powerful and direct methods have been developed for finding special solutions of nonlinear evolution equation(s), such as the inverse method [1], Backlund transformation [2], Hirota bilinear method [3], numerical methods [4], symmetry reduction [5], the Wronskian determinant technique [6], the tanh method [7], later various extended tanh methods [8–12], and the Lie group method [13,14], etc.

Discrete solitons in nonlinear lattices have been the focus of considerable attention in various branches of science. And many of them can be modelled by differential–difference equation(s) (DDE(s)). There is not too much work that has been done on many approaches in continuous nonlinear physical systems, for they can't be easily generalized to differential–difference systems.

Very recently, D. Baldwin et al. presented an algorithm [15] for finding exact traveling wave solutions of DDEs in terms of the tanh function and found kink-type solutions for many spatially discrete nonlinear models such as the hybrid lattice [14], Volterra lattice [15], discrete mKdV lattice [14,16], Ablowitz–Ladik lattice [1], Toda lattice [17]. And later, Xie extended the method. But Xie's [18] method is still unable to find solutions for polynomials in sech, cosech, cot, sec or cosec forms. Then, Zhu extend the exp function method to obtain rich solutions [19,20], while there are too many arbitrary constants in the expressions for the solutions. Inspired by their work, we newly introduced an effective algebraic method for finding soliton solutions in Ref. [21]. However, we were still unable to get periodic solutions.

In this paper, we would like to introduce a uniform method for constructing traveling wave solutions including periodic solutions and soliton solutions. The method is more general than that of Ref. [21].

2. Method and algorithm

Firstly, we would like to introduce some knowledge:

1. $\frac{d \sinh(\omega)}{d\omega} = \cosh(\omega)$, $\frac{d \cosh(\omega)}{d\omega} = \sinh(\omega)$, $\cosh^2(\omega) - \sinh^2(\omega) = 1$;
2. $\sinh(x \pm y) = \sinh(x) \cosh(y) \pm \cosh(x) \sinh(y)$, and $\cosh(x \pm y) = \cosh(x) \cosh(y) \pm \sinh(x) \sinh(y)$;

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3. $\sin(x \pm y) = \sin(x) \cos(y) \pm \cos(x) \sin(y)$, and $\cos(x \pm y) = \cos(x) \cosh(y) \mp \sin(x) \sin(y)$;
 4. by using the separation of variables method:
 if $\frac{d\omega}{d\xi} = \sinh(\omega)$, we can get $\sinh(\omega) = -\operatorname{cosech}(\xi)$, $\cosh(\omega) = -\coth(\xi)$;
 if $\frac{d\omega}{d\xi} = \cosh(\omega)$, we can obtain $\sinh(\omega) = -\cot(\xi)$, $\cosh(\omega) = \operatorname{cosec}(\xi)$.

Then, suppose that the NDDE that we discuss in this paper is in the following nonlinear polynomial form:

$$P(u_{n+p_1}(t), u_{n+p_2}(t), \dots, u_{n+p_s}(t), u'_{n+p_1}(t), u'_{n+p_2}(t), \dots, u'_{n+p_s}(t), \dots, u^{(r)}_{n+p_1}(t), u^{(r)}_{n+p_2}(t), \dots, u^{(r)}_{n+p_s}(t)) = 0, \quad (1)$$

where $u_n(t) = u(n, t)$ is a dependent variable; t is a continuous variable; $n, p_i, r \in \mathbb{Z}$.

Step 1. To compute the traveling wave solutions to Eq. (1), we first set $u_n(t) = u(\xi_n)$, and

$$\xi_n = dn + ct + \eta. \quad (2)$$

Namely, $\xi_{n+p_i} = (n + p_i)d + ct + \eta = \xi_n + dp_i$.

After the transformation (2), Eq. (1) can be rewritten as

$$G(u_{n+p_1}(\xi_n), u_{n+p_2}(\xi_n), \dots, u_{n+p_s}(\xi_n), u'_{n+p_1}(\xi_n), u'_{n+p_2}(\xi_n), \dots, u'_{n+p_s}(\xi_n), \dots, u^{(r)}_{n+p_1}(\xi_n), u^{(r)}_{n+p_2}(\xi_n), \dots, u^{(r)}_{n+p_s}(\xi_n)) = 0, \quad (3)$$

Step 2. Deriving the algebraic system.

In Ref. [21], we made the expansion

$$u_{n+p_i} = \sum_{i=-N}^N a_i \cosh^i(\omega_{n+p_i}) + \sum_{i=1}^N b_i \cosh^{i-1}(\omega_{n+p_i}) \sinh(\omega_{n+p_i}) + \sum_{i=-N}^{-1} c_i \cosh^i(\omega_{n+p_i}) \sinh(\omega_{n+p_i}), \quad (4)$$

with

$$\frac{d\omega_n}{d\xi_n} = \sinh(\omega_n), \quad (5)$$

$$\cosh(\omega_{n+p_i}) = -\coth(\xi_{n+p_i}) = \frac{\cosh(\omega_n) \cosh(p_i d) - \sinh(p_i d)}{\cosh(p_i d) - \cosh(\omega_n) \sinh(p_i d)}, \quad (6)$$

and

$$\sinh(\omega_{n+p_i}) = -\operatorname{cosech}(\xi_{n+p_i}) = \frac{\sinh(\omega_n)}{\cosh(p_i d) - \cosh(\omega_n) \sinh(p_i d)}. \quad (7)$$

However, the algorithm only constructed soliton solutions to DDE(s). We propose another form of u_{n+p_i} for seeking periodic solutions:

$$u_{n+p_i} = \sum_{i=-N}^N a_i \sinh^i(\omega_{n+p_i}) + \sum_{i=1}^N b_i \sinh^{i-1}(\omega_{n+p_i}) \cosh(\omega_{n+p_i}) + \sum_{i=-N}^{-1} c_i \sinh^i(\omega_{n+p_i}) \cosh(\omega_{n+p_i}), \quad (8)$$

with

$$\frac{d\omega_n}{d\xi_n} = \cosh(\omega_n), \quad (9)$$

$$\sinh(\omega_{n+p_i}) = -\cot(\xi_{n+p_i}) = \frac{\sin(dp_i) + \sinh(w_n) \cos(dp_i)}{\cos(dp_i) - \sinh(w_n) \sin(dp_i)}, \quad (10)$$

and

$$\cosh(\omega_{n+p_i}) = \operatorname{cosec}(\xi_{n+p_i}) = \frac{\cosh(w_n)}{\cos(dp_i) - \sinh(w_n) \sin(dp_i)}. \quad (11)$$

By introducing functions of f and g with parameter δ , we can combine (4) and (8) into a uniform framework:

$$u_{n+p_i} = \sum_{i=-N}^N a_i f^i(\omega_{n+p_i}) + \sum_{i=1}^N b_i f^{i-1}(\omega_{n+p_i}) g(\omega_{n+p_i}) + \sum_{i=-N}^{-1} c_i f^i(\omega_{n+p_i}) g(\omega_{n+p_i}), \quad (12)$$

with

$$\frac{d\omega_n}{d\xi_n} = g(\omega_n), \quad f_{n+p_i}^2 - g_{n+p_i}^2 = \delta, \quad (13)$$

$$g(\omega_{n+p_i}) = \frac{g(\omega_n)}{F(dp_i) - f(\omega_n)G(dp_i)}, \quad (14)$$

and

$$f(\omega_{n+p_i}) = \frac{f(\omega_n)F(dp_i) - \delta G(dp_i)}{F(dp_i) - f(\omega_n)G(dp_i)}. \quad (15)$$

where $\omega_n = \omega(\xi_n)$, and $a_0, a_{\pm 1} \dots a_{\pm N}, b_1 \dots b_N, c_{-1} \dots c_{-N}$ are unknown, to be determined later. N can be determined by balancing the highest degree linear term and nonlinear term of u_n .

Here $\delta^2 = 1$; in detail,

$$\begin{cases} f(\omega_n) = -\coth(\xi_n), & g(\omega_n) = -\operatorname{cosech}(\xi_n), & F(d) = \cosh(d), & G(d) = \sinh(d), & \text{when } \delta = 1; & \text{(a)} \\ f(\omega_n) = -\cot(\xi_n), & g(\omega_n) = \operatorname{cosec}(\xi_n), & F(d) = \cos(d), & G(d) = \sin(d), & \text{when } \delta = -1. & \text{(b)} \end{cases} \quad (16)$$

Then substituting (12) into Eq. (1), clearing the denominator, we obtain a finite series of $f^m(\omega_n)g^i(\omega_n)$ ($i = 0, 1 \dots m; m = 0, 1$). Setting the coefficients of $f^m(\omega_n)g^i(\omega_n)$ to zero, we get a set of algebraic equations with respect to the unknown a_i, b_i, c_i, k and c .

Step 3. We solve the nonlinear overdetermined algebraic equations, and we only need to substitute a_i, b_i, c_i, k and c by considering parameter δ .

3. Application of the algorithm

The mKdV lattice equation is

$$\frac{du_n(t)}{dt} = (\alpha - u_n^2(t))(u_{n+1}(t) - u_{n-1}(t)). \quad (17)$$

We seek for solutions to Eq. (7) in the framework

$$u_n = a_0 + \frac{a_{-1}}{f(\omega_n)} + a_1 f(\omega_n) + b_1 g(\omega_n) + c_{-1} \frac{g(\omega_n)}{f(\omega_n)}. \quad (18)$$

Substituting (8) with Eq. (4) into Eq. (7), clearing the denominator and setting the coefficients of all powers like $f^j(\omega_n)g^i(\omega_n)$ ($i = 0, 1 \dots m; j = 0, 1$) to zero yields some overdetermined algebraic equations with unknown $a_{-1}, a_0, a_1, b_1, c_{-1}, c$. To avoid tediousness, we omit the equations. And with the help of Maple, we obtain the following sets of solutions:

$$\text{Case 1. } a_{-1} = \frac{\delta G(d)}{F(d)} \sqrt{\alpha}, a_0 = a_1 = b_1 = c_{-1} = 0, k = \frac{2\alpha G(d)}{F(d)};$$

$$\text{Case 2. } c_{-1} = \sqrt{\frac{\alpha \delta}{\delta G^2(d) - F^2(d)}} G(d), a_0 = a_1 = a_{-1} = b_1 = 0, k = -\frac{2\alpha G(d)}{G^2(d)\delta - F^2(d)};$$

$$\text{Case 3. } a_1 = \frac{\sqrt{\alpha} F(d) G(d)}{G^2(d)\delta + F^2(d)}, a_{-1} = \frac{\delta \sqrt{\alpha} F(d) G(d)}{G^2(d)\delta + F^2(d)}, a_0 = b_1 = c_{-1} = 0, k = \frac{2F(d)G(d)\alpha}{G^2(d)\delta + F^2(d)};$$

$$\text{Case 4. } a_1 = \frac{\sqrt{\alpha} G(d)}{F(d)}, a_0 = a_{-1} = b_1 = c_{-1} = 0, k = \frac{2\alpha G(d)}{F(d)};$$

$$\text{Case 5. } a_1 = \frac{\sqrt{\alpha} F(d) G(d)}{-F^2(d) + G^2(d)\delta}, a_{-1} = \frac{\delta \sqrt{\alpha} F(d) G(d)}{F^2(d) - G^2(d)\delta}, a_0 = b_1 = c_{-1} = 0, k = \frac{2\alpha G(d)F(d)}{F^2(d) - G^2(d)\delta};$$

$$\text{Case 6. } b_1 = G(d) \sqrt{\frac{\alpha}{F^2(d) - G^2(d)\delta}}, a_0 = a_1 = a_{-1} = c_{-1} = 0, k = \frac{2\alpha G(d)}{F^2(d) - G^2(d)\delta}.$$

When $\delta = 1$, we can obtain six families of soliton solutions as follows:

$$\text{Family 1. } u_n = -\sqrt{\alpha} \tanh(d) \tanh(\xi_n), \xi_n = dn + 2\alpha \tanh(d)t + \eta;$$

$$\text{Family 2. } u_n = \sqrt{\alpha} i \sinh(d) \operatorname{sech}(\xi_n), \xi_n = dn + 2\alpha \sinh(d)t + \eta;$$

$$\text{Family 3. } u_n = -\frac{\sqrt{\alpha} \tanh(2d)}{2} \coth(\xi_n) - \frac{\sqrt{\alpha} \tanh(2d)}{2} \tanh(\xi_n), \xi_n = dn + \alpha \tanh(2d)t + \eta;$$

$$\text{Family 4. } u_n = -\sqrt{\alpha} \tanh(d) \coth(\xi_n), \xi_n = dn + 2\alpha \tanh(d)t + \eta;$$

$$\text{Family 5. } u_n = \frac{\sqrt{\alpha} \sinh(2d)}{2} \coth(\xi_n) - \frac{\sqrt{\alpha} \sinh(2d)}{2} \tanh(\xi_n), \xi_n = dn + \alpha \sinh(2d)t + \eta;$$

$$\text{Family 6. } u_n = -\sqrt{\alpha} \sinh(d) \operatorname{cosech}(\xi_n), \xi_n = dn + 2\alpha \sinh(d)t + \eta.$$

When $\delta = -1$, we can also obtain six seven families of periodic solutions in the following forms:

$$\text{Family 7. } u_n = \sqrt{\alpha} \tan(d) \tan(\xi_n), \xi_n = dn + 2\alpha \tan(d)t + \eta;$$

$$\text{Family 8. } u_n = -\sqrt{\alpha} \sin(d) \sec(\xi_n), \xi_n = dn + 2\alpha \sin(d)t + \eta;$$

$$\text{Family 9. } u_n = -\frac{\sqrt{\alpha} \tan(2d)}{2} \cot(\xi_n) + \frac{\sqrt{\alpha} \tan(2d)}{2} \tan(\xi_n), \xi_n = dn + \alpha \tan(2d)t + \eta;$$

$$\text{Family 10. } u_n = -\sqrt{\alpha} \tan(d) \cot(\xi_n), \xi_n = dn + 2\alpha \tan(d)t + \eta;$$

$$\text{Family 11. } u_n = \frac{\sqrt{\alpha} \sin(2d)}{2} \cot(\xi_n) + \frac{\sqrt{\alpha} \sin(2d)}{2} \tan(\xi_n), \xi_n = dn + \alpha \sin(2d)t + \eta;$$

$$\text{Family 12. } u_n = \sqrt{\alpha} \sin(d) \operatorname{cosec}(\xi_n), \xi_n = dn + 2\alpha \sin(d)t + \eta.$$

In the above twelve cases, η is arbitrary.

4. Summary

With symbolic computation, we have a uniform method for constructing soliton solutions and periodic solutions. On applying our method to Eq. (1), a rich variety of explicit solutions which include soliton solutions for sech, tanh, coth, cosech are obtained. Our method can also be used to find solutions of other DDE(s).

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